1800's: Part II

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Here we will take some shorter notes on the work done in the timespan 1850 - 1900 roughly. This is by the likes of Dedekind, Riemann, Lie and Hilbert. The work of Poincare and Cantor will be treated separately. Note that this period is also noteworthy because it is when all the great ideas of the previous fifty years got worked out in a more digestable fashion (looking at Galois here). Unfortunately the works are becoming longer, the authors more prolific and the mathematics more complicated. Im not sure how much I can really say, my thoughts become more opaque and confused the close we get to the present. I also think that the history is at this point mostly accurate.

1 Riemann (1854)

Translated with introduction [Jos25]. Here we have a true man of science. Its interesting to recount the story of how this paper came to be; Riemann was asked to present a talk for a job interview to a few people, but mainly Gauss, the topic that Gauss chose, for Riemann, was the foundations of geometry. Thus the influence of Gauss on this paper is evident, as Riemann references his work on curvature and indeed Gauss was the instigator.

This is a three fold work. There is a philosophical section in which he discusses the hypothesis that are necessary for geometry to take place. Once this is established he then defines a notion of curvature in this notion of geometry, that agrees with Gauss. Finally he discusses briefly applications of this idea to physics.

Riemann begins

> It is known that geometry assumes, as things given, both the notion of space and the first principles of constructions in space. She gives definitions of them which are merely nominal, while the true determinations appear in the form of axioms. The relation of these assumptions remains consequently in darkness; we neither perceive whether and how far their connection is necessary, nor, a priori, whether it is possible.

This is an interesting point as it emphasises that the real definition of the objects we work with are the axioms that we assign to them. Geometry is still thought of, at least as far as I can see, in the say way that Euclid did

> the most important system for our present purpose being that which Euclid has laid down as a foundation.

The first problem he attacks philosophically is "the definition of a multiply extended body". Despite referencing Euclid, he doesnt actually lay down new axioms, instead he just has a prolonged prose discussion about what he thinks is or is not necessary. In the absence of a measuring tool

> two magnitudes can only be compared when one is a part of the other ... magnitudes are regarded not as existing independently of position and not as expressible in terms of a unit, but as regions in a manifoldness.

he goes on to describe how to go from low dimensional manifolds to higher ones

> If in the case of a notion whose specialisations form a continuous manifoldness, one passes from a certain specialisation in a definite way to another, the specialisations passed over form a simply extended manifoldness, whose true character is that in it a continuous progress from a point is possible only on two sides, forwards or backwards. If one now supposes that this manifoldness in its turn passes over into another entirely different, and again in a definite way, namely so that each point passes over into a definite point of the other, then all the specialisations so obtained form a doubly extended manifoldness. etc

and from higher dimensional manifolds to lower ones. His first sentence means that a one dimensional manifold is anything that is given when passing continuously from one point to another. A two dimensional one is the manifold given by continuously dragging a one dimensional manifold over to another one.

Next he discusses the "the study of the measure-relations of which such a manifoldness is capable". It is here that he makes reference to Gauss.

> The hypothesis which first presents itself, and which I shall here develop, is that according to which the length of lines is independent of their position, and consequently every line is measurable by means of every other.

He develops a calculus on these manifolds in the style of Liebniz (infinitesimal tangents). He lays down some requirments on the differentials ds for a space to be flat, that is it has the form of a Euclidean metric (my words, not his).

Finally he remarks that the geomtery of the infinitely small and infinitely large may be non-flat in his sense, or non-Euclidean.

2 Dedekind (1872)

[DB63, I] (1872). Dedekinds motivation is that

> The statement is so frequently made that the differential calculus deals with continuous magnitude, and yet an explanation of this continuity is nowhere given ;

He claims that he struck apon his answer on "24th November 1858". He gives the analysis of the rational numbers as the collection of numbers generated by adding and multiplying units and then the inverse operation. Interestingly he, for the first time that I have seen, introduces a notations for *the collection of these numbers*, which he denotes R. He references his idea that this collection should be closed under the familiar operations. He describes how rational numbers correspond to the lengths of straight lines, however notes that

> Of the greatest importance, however, is the fact that in the straight line L there are infinitely many points which correspond to no rational number.

He gives the analysis that irrational numbers come from incomensurable lengths, but that for applications to arithmetic one should have an intrinsic definition coming from arithmetic. Now I disagree with his analisis, irrational numbers already had an intrinsic formulation via power series and continued fractions. I guess it is one of the miricles of mathematics that these ideas agree. Regardless, Dedekind analyses the line and concludes that

> The above comparison of the domain R of rational numbers with a straight line has led to the recognition of the existence of gaps, of a certain incompleteness or discontinuity of the former, while we ascribe to the straight line completeness, absence of gaps, or continuity.

and he beleives that the fundamental idea or definition of the continuity of the line is given by the property

> If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes, this severing of the straight line into two portions."

and indeed transporting this to the rationals leads directly to his definition of cuts.

It is interesting to note the presence of set theory in this work. Dedekind begins by referencing the work of Cantor,

> I am just in receipt of the interesting paper Ueber die Ausdehnung eines Satzes aus der Theorie der trigonotnetrischen Reihen, by G. Cantor, for which I owe the ingenious author my hearty thanks.

and says that they have come to the same conclusions on the definition of the reals, "But what advantage will be gained by even a purely abstract definition of real numbers of a higher type, I am as yet unable to see". We have seen already that he deals with the "collection" of all rationals and indeed the collection of all reals. His cuts are themselves collections of rational numbers.

Note that Dedekind defines his cuts and then states that

> we shall say that the number a corresponds to this cut, or that it produces this cut. From now on, therefore, to every definite cut there corresponds a definite rational or irrational number, and we regard two numbers as different or unequal always and only when they correspond to essentially different cuts.

so the real numbers *are not identified* with their cuts, but merely correspond to them and are identified with those other real numbers whose corresponding cuts are identified. There is an ontological difference, numbers are numbers and cuts are collections, but he defines one by a correspondence with the other. Note also that he just assumes lots of facts about sets, mostly forms of comprehension. So he has replaced geometric intuition with set theoretic intuition.

Remark. This paper also contains notions of limit that are very close to the modern one, although they are entirely prose. One should be aware that Frege was only active in the very late 19th century and that even still no one ever writes out such quantified statements (however they wouldnt use the prose alternatives "for every" etc either).

3 Hilbert (1899)

[HH94] (1899). It doesnt have any set theory but is an investigation into the axioms of geometry in the sense of Euclid. I could probably be interested in looking further and understanding how he proposed to prove certain things were mutually independent etc. But at this time I have lose steam for these things.

4 Reflections on this project

What I have been searching for in this investigation is honesty, clarity and precident. The words that escaped me at the outset were *paradigm shifts*, I was most interested in seeking out those works or periods in which new concepts or ways of doing mathematics emerged. I was not so interested in the proofs or theorems of the works but the underlying concepts and ideas that are now common. This is in addition to looking for the use of axiomatics and ontology.

The works that I read were those that I thought contained paradigm shifts, or clarifications of recent paradigm shifts. For these works Euclid was the starting point, and he had a clear emphasis on the deductive method, definitions and axioms. The next writer that seems to have paid much attention to axiomatics was Riemann in the 1850's. Writers inbetween seem to have been aware of Euclids axioms at times and vaguely have adopted them, as their foundations. Arguments were rarely precise in the way that Euclid was. Axioms only mentioned in passing, as in Descarte saying he assumes more general measuring instruments than Euclid. Riemann points out that the definitions of Euclid were vacuous only serving to give the name, while it was the axioms and postulates that actually described and "defined" the thing. Unfortunely later mathematicians only performed the first of the two steps, giving names but not properties that could actually be employed. Now it is possible that my sampling is biased, I think Dedekind remarked that to the discoverer all routes are permissable, but it is for posterity to form a clear description and logical route. What this means is that perhaps authors other than those that I have read, but contemporaries, were more concerned with axiomatics and clear definitions. Perhaps not (I suspect this). **Room for future reading I suppose.**

On the other hand the writers that I read were profound in their introduction of objects of distinct ontological status. Most of the shifts came from either the invention of new objects to study or the connection of old ones (which in itself is the invention of the connecting object). Many of these objects have themselves now fallen out of use, being replaced with higher or more abstract substitutes. Connecting geometry and number theory, the introduction of infinitesimals, functions, infinite series, and groups or collections of objects. Perhaps it was the introduction of such things that constituted the paradigm shift.

One thing I noticed is that paradigm shifts, despite being what I searched for, rarely happened. Mathematics was, and perhaps is, extremely gradual in its adoption of new concepts. Their introduction is usually hinted at, scattered throughout many different authors, before one author takes the initiative to treat the subject "in itself". For instance Fermats notes on calculus circulated for years, before Liebniz provided (at least to me) a more succinct and clear work on it, but even Liebniz work was far from complete, with it waiting for people like Euler to make it work more generally for curves coming from arbitrary functions, and even after this the foundations were not settled, as there remained fierce debate about what it all really meant.

One thing that shocked me is how truly modern most of mathematics is. The 18th century is when we developed the calculus curriculum. The 19th century is where essentially mathematics began and if we are honest with ourselves, really only in the second half of that century, and really to be more honest mainly in the 20th century. Before 1850 mathematics was mainly still in the paradigm of Euclid, there was number theory and geometry and the great labour had been performed of clarifying their connection. The only other of the various mathematical objects that we now have that existed clearly was that of a function. In 1850 - 1900 things like groups, sets and rings were developed abstractly (although not in the absolutely extremely abstract form that we now have them (yes even set theory was much more concrete)). All other modern mathematical objects or concepts are from the 20th century as far as we can tell, and these objects were so altered by the intermediate time it is fair also to attribute them to later as well. Things like sets of functions, topological spaces, abstract vector spaces, groups or rings (their elements being arbitrary) are from the 20th century.

References

[DB63] Richard Dedekind and Wooster Woodruff Beman. *Essays on the theory of numbers.* Dover books on advanced mathematics. Dover, New York, 1963.

[HH94] David Hilbert and David Hilbert. *The foundations of geometry.* Open Court, La Salle, Ill, 2. engl. ed., [nachdr.] edition, 1994.

[Jos25] Jürgen Jost. *Bernhard Riemann — On the Hypotheses Which Lie at the Bases of Geometry.* Classic Texts in the Sciences. Springer Nature Switzerland, Cham, 2025.